NON COMPLETE AFFINE CONNECTIONS ON FILIFORM LIE ALGEBRAS

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Abstract. We give a example of non nilpotent faithful representation of a filiform Lie algebra. This gives one counter-example of the conjecture saying that every affine connection on a filiform Lie group is complete.

1. Affine connection on a nilpotent Lie algebra

1.1. Affine connection on nilpotent Lie algebras.

Definition 1. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra over $\mathbb{R}$. It is called affine if there is a bilinear mapping*

$$\nabla : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying

$$\begin{cases} 1) & \nabla(X,Y) - \nabla(Y,X) = [X,Y] \\ 2) & \nabla(X,\nabla(Y,Z)) - \nabla(Y,\nabla(X,Z)) = \nabla([X,Y],Z) \end{cases}$$

for all $X,Y,Z \in \mathfrak{g}$.

If $\mathfrak{g}$ is affine, then the corresponding connected Lie group G is an affine manifold such that every left translation is an affine isomorphism of G. In this case, the operator ∇ is nothing that the connection operator of the affine connection on G.

Let $\mathfrak{g}$ be an affine Lie algebra. Then the mapping

$$f : \mathfrak{g} \rightarrow End(\mathfrak{g})$$

defined by

$$f(X)(Y) = \nabla(X,Y)$$

is a linear representation (non faithful) of $\mathfrak{g}$ satisfying

$$f(X)(Y) - f(Y)(X) = [X,Y] \qquad (*)$$

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Remark. *The adjoint representation $\widetilde{f}$ of $\mathfrak{g}$ satisfies*

$$\widetilde{f}(X)(Y)-\widetilde{f}(Y)(X)=2\,[X,Y]$$

and cannot correspond to an affine connection.

1.2. Classical examples of affine connection.

i) Let $\mathfrak{g}$ be the n-dimensional abelian Le algebra. Then the representation

$$f:\mathfrak{g}\rightarrow End(\mathfrak{g})$$

given by

$$X\mapsto f(X)=0$$

defines an affine connection.

ii) Let $\mathfrak{g}$ be an $2p$-dimensional Lie algebra endowed with a symplectic form :

$$\theta\in\Lambda^2\mathfrak{g}^* \text{ such that } d\theta=0$$

with

$$d\theta(X,Y,Z)=\theta(X,[Y,Z])+\theta(Y,[Z,X])+\theta(Z,[X,Y]).$$

For every $X\in\mathfrak{g}$ we can define an unique endomorphism f_X by

$$\theta(adX(Y),Z)=-\theta(Y,f_X(Z)).$$

Then $\nabla(X,Y)=f_X(Y)$ is an affine connection.

iii) Following the work of Benoist [1], we know that exists nilpotent Lie algebra without affine connection.

1.3. Faithful representations associated to an affine connection.

Let ∇ be an affine connection on n-dimensional Lie algebra $\mathfrak{g}$. Let us consider the $(n+1)$-dimensional linear representation given by

$$\rho:\mathfrak{g}\rightarrow\mathfrak{g}\oplus\mathbb{R}$$

given by

$$\rho(X):(Y,t)\mapsto(f_X(Y)+tX,0)$$

It is easy to verify that ρ is a faithful representation of dimension $n+1$ if and only if $f_X(Y)=\nabla(X,Y)$ is an affine connection.

Definition 2. *We say that the representation ρ is nilpotent if the endomorphism $\rho(X)$ is nilpotent for every X in $\mathfrak{g}$.*

Proposition 1. *Suppose that $\mathfrak{g}$ is a complex nilpotent Lie algebra and let ρ be a faithful representation of $\mathfrak{g}$. Then there exists a faithful nilpotent representation.*

Proof: Let us consider the $\mathfrak{g}$-module M associated to ρ. Then, as $\mathfrak{g}$ is nilpotent, M can be decomposed as

$$M=\oplus_{i=1}^{k}M_{\lambda_i}$$

where M_{λ_i} is a $\mathfrak{g}$-submodule, and the λ_i are linear forms on $\mathfrak{g}$. For all $X \in \mathfrak{g}$, the restriction of $\rho(X)$ to M_i as the following form

$$\begin{pmatrix} \lambda_i(X) & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_i(X) \end{pmatrix}$$

Let $\mathbb{C}_{\lambda_i}$ be the one dimensional $\mathfrak{g}$-module defined by

$$\mu : X \in \mathfrak{g} \to \mu(X) \in End\mathbb{C}$$

with

$$\mu(X)(a) = \lambda_i(X)a$$

The tensor product $M_{\lambda_i} \otimes \mathbb{C}_{-\lambda_i}$ is the $\mathfrak{g}$-module associated to

$$X \cdot (Y \otimes a) = \rho(X)(Y) \otimes a - Y \otimes \lambda_i(X)a$$

Then $\widetilde{M} = \oplus (M_{\lambda_i} \otimes C_{-\lambda_i})$ is a nilpotent $\mathfrak{g}$-module. Let us prove that $\widetilde{M}$ is faithful. Recall that a representation ρ of $\mathfrak{g}$ is faithful if and only if $\rho(Z) \neq 0$ for every $Z \neq 0 \in Z(\mathfrak{g})$. Consider $X \neq 0 \in Z(\mathfrak{g})$. If $\widetilde{\rho}(X) = 0$, the endomorphism $\rho(X)$ is diagonal. Suppose that $\mathfrak{g} \neq Z(\mathfrak{g})$ and let $\mathcal{C}^{k-1}(\mathfrak{g}) = Z(\mathfrak{g})$ where k is the index of nilpotence of $\mathfrak{g}$. Then

$$\exists\, (Y, Z) \in \left(\mathcal{C}^{k-2}(\mathfrak{g}), \mathfrak{g}\right) \diagup [Y, Z] = X$$

The endomorphism $\rho(Y)\rho(Z) - \rho(Z)\rho(Y)$ is nilpotent and the eigenvalues of $\rho(X)$ are 0. Thus $\rho(X) = 0$ and ρ is not faithful. We can conclude that $\widetilde{\rho}(X) \neq 0$ and $\widetilde{\rho}$ is a faithful representation.

2. Affine connection on filiform Lie algebra

2.1. Definition.

Definition 3. *A n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called filiform if the smallest k such that $\mathcal{C}^k\mathfrak{g} = \{0\}$ is equal to $n-1$.*

In this case the descending sequence is

$$\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \cdots \supset \mathcal{C}^{n-2}\mathfrak{g} \supset \{0\} = \mathcal{C}^{n-1}\mathfrak{g}$$

and we have
$$\begin{cases} \dim \mathcal{C}^1\mathfrak{g} = n-2, \\ \dim \mathcal{C}^i\mathfrak{g} = n-i-1, \text{ for } i = 1, ..., n-1. \end{cases}$$

Example. *The n-dimensional nilpotent Lie algebra L_n defined by*

$$[X_1, X_i] = X_{i+1} \text{ for } i \in \{2, ..., n-1\}$$

is filiform.

We can note that any filiform Lie algebra is a linear deformation of L_n [6].

2.2. On the nilpotent affine connection.

Let $\mathfrak{g}$ be a filiform affine Lie algebra of dimension n, and ρ be the $(n+1)$-dimensional associated faithful representation. Let $M = \mathfrak{g} \oplus \mathbb{C}$ be the corresponding complex $\mathfrak{g}$-module. As $\mathfrak{g}$ is filiform, its decomposition has the following form

$$\begin{array}{l} 1)\ M = M_0 \quad \text{and } M \text{ is irreducible,} \\ 2)\ M = M_0 \oplus M_\lambda. \end{array}$$

For a general faithful representation, let us call characteristic the ordered sequence of the dimensions of the irreducible submodules. In the filiform case we have $c(\rho) = (n+1)$ or $(n, 1)$. In fact, the filiformity of $\mathfrak{g}$ implies that exists an irreducible submodule of dimension greater than $n-1$. More generally, if the characteristic sequence of a nilpotent Lie algebra is equal to $(c_1, .., c_p, 1)$ (see [6]) then for every faithful representation ρ we have $c(\rho) = (d_1, .., d_q)$ with $d_1 \geq c_1$.

Theorem 1. *Let $\mathfrak{g}$ be the filiform Lie algebra L_n. There are faithful $\mathfrak{g}$-modules which are not nilpotent.*

Proof: Consider the following representation given by the matrices $\rho(X_i)$ where $\{X_1, .., X_n\}$ is a basis of $\mathfrak{g}$

$$\rho(X_1) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & & 0 & 1 \\ a & a & 0 & & & & & \vdots & 0 \\ 0 & 0 & 0 & & & & & 0 & 0 \\ \vdots & \ddots & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{i-3}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \alpha & \beta & 0 & \cdots & \cdots & 0 & \frac{n-3}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(X_2) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & \cdots & 0 & 0 \\ a & a & 0 & & & & & \vdots & 1 \\ -1 & 1 & 0 & & & & & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{1}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \beta & \alpha & 0 & \cdots & \cdots & \cdots & \frac{1}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and for $3 \leq j \leq n-1$ the endomorphismes $\rho(X_j)$ satisfy :

$$\begin{cases} \rho(X_j)(e_1) = -\dfrac{1}{j-1}e_{j+1} \\ \rho(X_j)(e_2) = \dfrac{1}{j-1}e_{j+1} \\ \rho(X_j)(e_3) = \dfrac{1}{j(j-1)}e_{j+2} \\ \cdots \\ \rho(X_j)(e_{i-j+1}) = \dfrac{(j-2)!(i-j-1)!}{(i-2)!}e_i, \quad i = j-2, ..., n \\ \rho(X_j)(e_{i-j+1}) = 0, \quad i = n+1, .., n+j-1 \\ \rho(X_j)(e_{n+1}) = e_j \end{cases}$$

and for $j = n$

$$\begin{cases} \rho(X_n)(e_i) = 0 \quad i = 1, ..., n \\ \rho(X_n)(e_{n+1}) = e_n \end{cases}$$

where $\{e_1, ..., e_n, e_{n+1}\}$ is the basis given by $e_i = (X_i, 0)$ and $e_{n+1} = (0, 1)$. We easily verify that these matrices describe a non nilpotent faithful representation.

2.3.Study of an associated connection.

The previous representation is associated to an affine connection on the filiform Lie algebra L_n given by

$$\nabla_{X_i} = \rho(X_i) \mid_{\mathfrak{g}}$$

where $\mathfrak{g}$ designates the $n-$dimensional first factor of the $(n+1)-$ dimensional faithful module. This connection is complete if and only if the endomorphismes $R_X \in End(\mathfrak{g})$ define by

$$R_X(Y) = \nabla_Y(X)$$

are nilpotent for all $X \in \mathfrak{g}$ ([5]). But the matrix of R_{X_1} has the form :

$$\begin{pmatrix} a & a & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a & a & & & \vdots & & \vdots & 0 \\ 0 & -1 & & & \vdots & & \vdots & 0 \\ 0 & 0 & -\frac{1}{2} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & & \cdots & \vdots & 0 \\ 0 & 0 & \vdots & \ddots & -\frac{1}{j-1} & & \vdots & 0 \\ \alpha & \beta & \vdots & \cdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n-2} & 0 \end{pmatrix}$$

Its trace is $2a$ and for $a \neq 0$ it is not nilpotent. We have proved :

Proposition 2. *There exist affine connexions on the filiform Lie algebra L_n which are non complete.*

Remark. *The most simple example is on dim*3 *and concerns the Heisenberg algebra. We find a nonnilpotent faithful representation associated to the noncomplete affine connection given by :*

$$\nabla_{X_1}=\begin{pmatrix} a & a & 0\\ a & a & 0\\ \alpha & \beta & 0\end{pmatrix},\quad \nabla_{X_2}=\begin{pmatrix} a & a & 0\\ a & a & 0\\ \beta-1 & \alpha+1 & 0\end{pmatrix},\quad \nabla_{X_3}=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The affine representation is written

$$\begin{pmatrix} a(x_1+x_2) & a(x_1+x_2) & 0 & x_1\\ a(x_1+x_2) & a(x_1+x_2) & 0 & x_2\\ \alpha x_1+(\beta-1)x_2 & \beta x_1+(\alpha+1)x_2 & 0 & x_3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

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